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On Projective Dimension of Spline Modules

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Let a region Ω of the euclidean space $\mathbf{R}^d$ ($d \geqslant 1$) be decomposed as a polyhedral complex $\square$, and let $S^r(\square)$ denote the set of all multivariate c^r-splines on $\square$. Then, with pointwise operations, the set $S^r(\square)$ turns out to be a finitely generated torsion free module over the ring $R = \mathbf{R}[x_1, ..., x_d]$ of polynomials in d variables. In this paper, the results of Billera and Rose on the freeness of this R-module on triangulated regions are extended to the projective dimension of this module and on arbitrary polygonal subdivisions. Possible relationships between the projective dimensions of the spline modules on subcomplexes have been established. Examples illustrating the theorems and counterexamples limiting the possibilities have been presented. In particular, an example showing that freeness of the spline module $S^r(\square)$ is not a local concept for general polyhedral complexes, as against the triangulated ones, has been constructed. © 1996 Academic Press, Inc.

1. INTRODUCTION

Let a region Ω of the plane $\mathbf{R}^2$ be subdivided into a finite set of smaller pieces like triangles, rectangles, or any polygonal units so that their union is Ω, and the intersection of any two of the pieces is a one-dimensional face of both. Such a subdivision, known as polyhedral decomposition of Ω, can be viewed as a nice approximation to the region Ω itself. Real valued functions defined on Ω, which are piecewise polynomials in two variables on such a decomposition, have proved to be extremely useful in obtaining the desired approximations to the solutions of partial differential equations by finite element methods. These functions, generally known as splines, can be easily manipulated on computers, and therefore, readily fulfil the computational needs of approximations very efficiently. Algebraic and analytical properties of splines are, by and large, quite similar to those of polynomials. However, there are situations required by approximation theorists where the behaviour of splines is even better than polynomials. As a result, multivariate splines including the univariate ones, have found enormous

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applications in engineering sciences, and more recently in computer aided geometric designs and computer graphics.

If $\square$ denotes the polyhedral decomposition of a region Ω in $\mathbf{R}^2$, and $S_k^r(\square)$ denotes the set of all splines on Ω which are r times $(r \geqslant 0)$ continuously differentiable on Ω and are of degree at most k $(\geqslant 1)$, then with pointwise operations of addition and scalar multiplication, the set $S_k^r(\square)$ is a vector space over $\mathbf{R}$. Determining the dimension of this vector space in terms of r, k, and other structures of $\square$, and constructing a nice basis with minimal supports, is known as the "dimension problem" of multivariate splines (univariate case is easy and completely known). The observation that this dimension depends not only on the combinatorial invariants, but on the geometry of $\square$ also, is one of the main reasons of the complexity and depth of the dimension problem which is still far from resolved. Knowing computable dimension formulae would obviously facilitate the computation of sharper approximation results. By now several methods, including the homological approach of Louis Billera (see [2]), have been attempted to solve this problem. In order to tackle it further and to have deeper understanding about the class of these valuable spline functions, it now seems necessary to look at other available algebraic structures on them. Billera's solution of the Strang's conjecture (Theorem 5.8, p. 337 of [2]) is a convincing illustration of such a study. It is on these lines that the objective of this paper is set forth essentially to expand and carry further such an algebraic and combinatorial approach initiated by Billera and Rose. This is done in the most general setting of arbitrary multivariate (not just bivariate) splines on any polyhedral decomposition of a region embedded in the d-dimensional euclidean space $\mathbf{R}^d$ $(d \geqslant 1)$.

Thus let $\square$ be a polyhedral d-complex (see preliminaries) embedded in $\mathbf{R}^d$ and consider the set $S^r(\square)$ of all piecewise polynomials on $\square$ which are continuously differentiable of order a given $r \geqslant 0$ on whole of $\square$. With respect to pointwise operations of addition and multiplication, the set $S^r(\square)$ is a ring and the set $R = \mathbf{R}[x_1, ..., x_d]$ considered as global polynomials on $\square$ forms a subring of $S^r(\square)$. Hence $S^r(\square)$ is naturally an R-module and is called the spline module of c^r-splines on $\square$. In a series of papers (see [5], [6], [7]), Billera and Rose initiated the study of this module and obtained some basic results. For example, in [6] and [7], the methods of commutative algebra have been used to study the dimension problem (mentioned earlier) of $\mathbf{R}$-vector spaces $S_k^r(\triangle)$ of all c^r-splines on $\triangle$ of degree at most k when $\square = \triangle$ is a simplicial complex; in [5] the algebraic question viz. under what conditions on $\square$, r, and d the R-module $S^r(\square)$ would be free, was studied; the case $d = 2$ was completely solved ($d = 1$ being trivial) for all polyhedral complexes $\square$, and the case $r = 0$ was solved for all d but for simplicial complexes $\triangle$ only. In both of these cases, it follows from their results that freeness of the R-module $S^r(\square)$ depends on the topology of the union of faces of $\square$ rather than on the geometry of $\square$ unlike the case for $\dim_{\mathbf{R}} S_k^r(\triangle)$, $r>0$. In [12], Yuzvinsky considered the projective dimension of the module $S^r(\square)$ (projective dimension zero being the free case) and obtained interesting generalization of Billera–Rose results valid for arbitrary polyhedral complexes $\square$. More precisely, Yuzvinsky has obtained a characterization of the projective dimension of $S^r(\square)$ in terms of the sheaf cohomology of certain subsets of a canonically associated poset L of $\square$ with coefficients in a sheaf of R-modules (see Section 3) defined on L. The first example of a polyhedral complex $\square$ for which the freeness of even $S^0(\square)$ depends on the geometry of $\square$, was given by Billera (unpublished). However, Yuzvinsky, using his criterion of the projective dimension of $S^0(\square)$, provides yet another example of such a complex. These are in remarkable contrast with the case when $\square=\triangle$ is a simplicial complex where the freeness of $S^0(\triangle)$ is indeed a topological property [5].

In their study of the freeness of the spline module $S^r(\square)$, Billera–Rose ([5], section 2) have shown that if $\square=\triangle$ is a simplicial complex, then for $r\geqslant 0$, $S^r(\triangle)$, is free over R iff $S^r(Stv)$ is free over R for each vertex v of $\triangle$. Since projective modules over $R=\mathbf{R}[x_1, ..., x_d]$ are free, this says that projective dimension of $S^r(\triangle)$, being zero, is indeed a local concept. As remarked by them their proof is valid for simplicial complexes only. In this paper, we generalize their result and show that projective dimension, being less than or equal to n, is a local concept for any $n\geqslant 0$. We also show that such a result is not true for polyhedral complexes, in general.

Our main concern, however, is to deal with the following question: If $\square'$ is a d-subcomplex of $\square$, when can we say that $pd_R S^r(\square')\leqslant pd_R S^r(\square)$? This is like asking as to when is the "subset theorem" of classical dimension functions of topology valid for the projective dimension of spline modules. In Section 3, we will identify a wide class of d-subcomplexes for which the monotonicity of the projective dimension can be asserted. Section 2 is devoted to needed preliminaries. In Section 4 we look at the analogue of "sum theorems" for the projective dimension, and prove that for simplicial complexes $\triangle$, the projective dimension of $S^r(\triangle)$ is a local concept, i.e., it is completely determined by the projective dimension of $S^r(Stv)$ where v runs over all vertices of $\triangle$. Finally, we give an example to show that the projective dimension of $S^r(\square)$ for general polyhedral complexes $\square$ need not be a local concept.

2. Preliminaries

We recall (see [5]) that a finite collection $\square$ of convex polyhedra in $\mathbf{R}^d$ is called a polyhedral complex if (i) any face of a member of $\square$ is a member of $\square$, and (ii) the intersection of any two members of $\square$ is a face of both. A simplicial complex is thus a special case of a polyhedral complex when all faces of $\square$ are simplexes. Just to distinguish the simplicial case from the general polyhedral case we will use the symbol $\triangle$ for a simplicial complex. We will identify $\square$ with the union of members of $\square$ which is a subset of $\mathbf{R}^d$. With this understanding the collection $\square$ (resp. $\triangle$) is called a polyhedral decomposition (resp. triangulation) of the region $\square$ (resp. $\triangle$) of $\mathbf{R}^d$. If every maximal member of $\square$ is of the same dimension d, we will say that $\square$ is a d-complex in $\mathbf{R}^d$. The polynomial ring $\mathbf{R}[x_1, ..., x_d]$ over the field of real numbers in d indeterminates will be denoted by R unless stated otherwise. If σ is a face of $\square$, aff(σ) will denote the affine subspace of $\mathbf{R}^d$ generated by σ, and $I(\sigma) = I(\mathrm{aff}(\sigma))$ will denote the ideal of all polynomials in R which vanish on σ. The symbol $(I(\sigma))^r$ will denote the r-fold product of the ideal $I(\sigma)$ (see [5] for details).

Now let $\square$ be a polyhedral decomposition of a region Ω in the space $\mathbf{R}^d$. To say that $f \in S^r(\square)$ means, for each maximal face σ of $\square$, $f \mid \sigma$ is a polynomial in d real variables $x_1, ..., x_d$, and f is continuously differentiable r times on $\square$, i.e., all partial derivatives of f of order $\leqslant r$ exist and are continuous at every point of $\square$. If x is an interior point of some maximal face σ, then evidently f is given by a polynomial in a small neighbourhood of x in Ω and hence f is even analytic at x. Care must be taken only when x lies in the common boundary of two or more maximal faces. In that case, if x belongs to two faces, say σ, σ', then f at x is given by two polynomials $f \mid \sigma$ and $f \mid \sigma'$, and the c^r-condition requires that all partial derivatives of f computed by taking $f \mid \sigma$ or $f \mid \sigma'$ at x must be identical. Billera and Rose have shown [6] that this analytic condition is equivalent to the following algebraic condition:

Algebraic criterion. A piecewise polynomial f on $\square$ is in class c^r iff for every pair of maximal faces σ, σ' in $\square$

$$f \mid \sigma - f \mid \sigma' \in (I(\sigma \cap \sigma'))^{r+1}.$$

We will frequently use this later on. Let us also recall that if the d-complex $\square$ has t maximal faces, then by fixing an ordering, say $\sigma_1, \sigma_2, ..., \sigma_t$ on them, a spline $f \in S^r(\square)$ can be thought of as a t-tuple $(f_1, ..., f_t)$ of polynomials $f_i = f \mid \sigma_i$ in the ring $R = \mathbf{R}[x_1, ..., x_d]$ satisfying the algebraic condition stated above. Consequently, the ring $S^r(\square)$ is a subring of the product ring R^t, and also an R-submodule of the R-module R^t. By Hilbert basis theorem, R is noetherian; it is a P.I.D. iff $d = 1$. It can now be shown that the spline module $S^r(\square)$ over R is finitely generated, torsion free and of rank r [6]. Any projective module over the polynomial ring R is free, and similarly a projective module over a local ring is also free (e.g. see [8]).

A property Q of an A-module M is said to be a local property if M has Q iff for each prime ideal p of A, the localized module M_p has the property Q over the ring A_p (see [1] p. 40 for details and examples). The following well-known result asserts that to be projective is a local property of finitely generated modules over noetherian rings (cf. Matsumura: Commutative Algebra, Theorem 7.12):

PROPOSITION 2.1. *Let M be a finitely generated A-module where A is noetherian. Then the following are equivalent*:

(i) *M is projective over A.*

(ii) *M_p is projective over A_p for every prime ideal p of A.*

(iii) *M_m is projective over A_m for every maximal ideal m of A.*

The projective dimension of an A-module M over A will be denoted by $pd_A M$. It is well known that for any $n \geqslant 0$, $pd_A M \leqslant n$ iff given any resolution

$$0 \to K_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

of M where P_i are projective over A, $i = 0, 1, ..., n-1$, implies that K_n is also projective. Using this criterion and the above Proposition one can easily deduce (*see Corollary* 1.6, *p.* 199 *of* [8]).

PROPOSITION 2.2. *Let A be a noetherian ring and M be a finitely generated A-module. Then, for any $n \geqslant 0$,*

$$pd_A M \leqslant n \qquad \textit{iff} \quad pd_{A_p} M_p \leqslant n$$

for each maximal (*resp. prime*) *ideal p of A.*

The following result will be frequently used in the sequel:

COROLLARY 2.1. *Let $r \geqslant 0$ and $\square$ be any polyhedral d-complex embedded in $\mathbf{R}^d$. Then for any $n \geqslant 0$, $pd_R S^r(\square) \leqslant n$ iff $pd_{R_m}(S^r(\square))_m \leqslant n$ for every maximal ideal* (*respectively prime ideal*) *m of R.*

3. PROJECTIVE DIMENSION OF SUBCOMPLEXES

Let $\square$ be a d-complex embedded in $\mathbf{R}^d$. A d-complex $\square'$ embedded in $\mathbf{R}^d$ is said to be a d-subcomplex of $\square$ if $\square'$ is a subcomplex of $\square$. In this section, we wish to consider those d-subcomplexes $\square'$ of $\square$ which satisfy the condition that $pd_R S^r(\square') \leqslant pd_R S^r(\square)$—we will express this by saying that

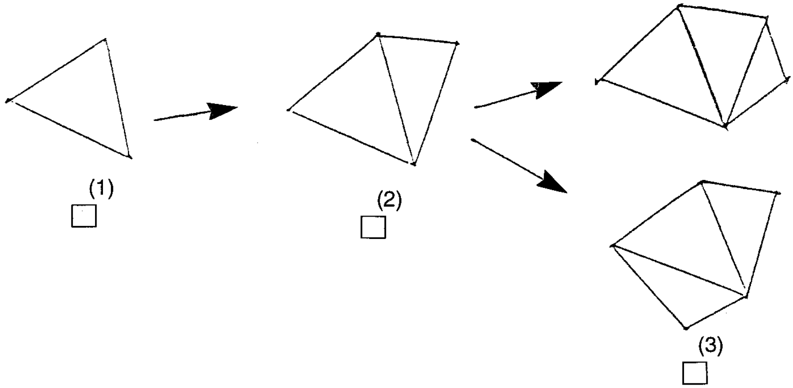

FIGURE 1

$\square'$ satisfies the subset theorem for projective dimension of spline modules. $S^r(\square')$ need not be a submodule of $S^r(\square)$, and even if it is, we don't expect any relationship between the projective dimensions of $S^r(\square')$ and $S^r(\square)$ as such. We give examples (see Example 3.1) to show that all possibilities can occur. However, one can still identify a class of d-subcomplexes $\square'$ of $\square$ for which the subset theorem can be proved.

Consider a d-face τ of a d-complex $\square$. Suppose there is a d-subcomplex $\square'$ of $\square$ such that $\square' \cap \tau$ is a $(d-1)$-face of τ; then we will say that $\square'' = \square' \cup \tau$ has been obtained from $\square'$ by attaching τ along a $(d-1)$-face of $\square$. More generally, let $\square^{(1)} \subset \cdots \subset \square^{(k)}$ be a sequence of d-subcomplexes of $\square$ such that for each $i = 1, 2, ..., k-1$, $\square^{(i+1)}$ has been obtained from $\square^{(i)}$ by attaching a d-face of $\square$ along a $(d-1)$-face. Then we will say that $\square^{(k)}$ has been obtained from $\square^{(1)}$ by attaching a sequence of d-faces along $(d-1)$ faces. Note that the growth from $\square^{(1)}$ to $\square^{(k)}$ can take place horizontally as well as vertically or both. For instance, $\square^{(3)}$ (case $d = 3$) can be obtained from $\square^{(2)}$ by attaching a d-face along a $(d-1)$-face of $\square^{(2)}$ which may or may not be a face of $\square^{(1)}$ (Fig. 1).

One of our main results (Theorem 3.2) to be proved later, will assert that if starting from the star of a face of a d-complex $\square$, a d-subcomplex $\square'$ of $\square$ is obtained by successively attaching a finite number of d-faces along $(d-1)$-faces (both integers d as well as $(d-1)$ as mentioned, are crucial to the validity of the result), then the projective dimension of the spline module of $\square'$ will never exceed that of $\square$ for any given degree of smoothness.

Remark 3.1. It may be observed that any two d-faces of $\square^{(k)}$ are connected by a sequence of d-faces of $\square^{(k)}$ itself, i.e., the d-complex $\square^{(k)}$ has a connected graph.

First let us prove

PROPOSITION 3.1. *Let $\square$ be a d-complex embedded in $\mathbf{R}^d$ and suppose $\square'$ is obtained from $\square$ by attaching a sequence of d-faces along $(d-1)$ faces. Then for every $r \geqslant 0$*

$$pd_R S^r(\square') = pd_R S^r(\square).$$

In particular, if $S^r(\square)$ is free, then so is $S^r(\square')$.

Proof. By induction on the number of terms in the sequence; and obviously it suffices to prove the case when $\square'$ is obtained from $\square$ by just attaching one d-face, say σ along a $(d-1)$-face $\sigma' \in \square$. For convenience, we order d-faces of $\square'$ so that σ' and σ occur at the end, and index the components of spline on $\square'$ by the d-faces themselves. First, we claim that $(f_{\sigma_1}, ..., f_{\sigma'}, f_\sigma) \in S^r(\square')$ iff $(f_{\sigma_1}, ..., f_{\sigma'}) \in S^r(\square)$ and $f_\sigma - f_{\sigma'} \in (I(\sigma \cap \sigma'))^{r+1}$. We have only to prove the converse and it suffices to show that for each $\sigma_i \in \square$, $f_\sigma - f_{\sigma_i} \in (I(\sigma \cap \sigma_i))^{r+1}$. Since $\sigma_i \cap \sigma \subset \sigma'$ and $\sigma_i \cap \sigma \subset \sigma_i$, $\sigma_i \cap \sigma \subset \sigma_i \cap \sigma'$, so that $(I(\sigma_i \cap \sigma))^{r+1} \supseteq (I(\sigma_i \cap \sigma'))^{r+1}$. Also, $\sigma_i \cap \sigma \subset \sigma' \cap \sigma$, and so we have

$$\begin{aligned} f_\sigma - f_{\sigma_i} &= f_\sigma - f_{\sigma'} + f_{\sigma'} - f_{\sigma_i} \\ &\in (I(\sigma \cap \sigma')^{r+1} + (I(\sigma' \cap \sigma_i))^{r+1} \\ &\subseteq (I(\sigma_i \cap \sigma))^{r+1}. \end{aligned}$$

Next, let us define an R-map $\theta\colon S^r(\square) \to S^r(\square')$ defined by

$$\theta(f_{\sigma_1}, ..., f_{\sigma'}) = (f_{\sigma_1}, ..., f_{\sigma'}, f_{\sigma'}).$$

By what we proved above, θ is well defined and admits an R-splitting $\psi\colon S^r(\square') \to S^r(\square)$ defined by

$$\psi(f_{\sigma_1}, ..., f_{\sigma'}, f_\sigma) = (f_{\sigma_1}, ..., f_{\sigma'}).$$

Hence as R-modules,

$$S^r(\square') \cong S(\square) \oplus Ker\psi.$$

Since $Ker\psi$ consists of all elements of $S^r(\square')$ of the form $(o, o, ..., f_\sigma)$, we find that $Ker\psi \cong R$. ∎

Note that it follows from the above result that whenever $\square'$ is obtained from $\square$ by attaching a sequence of d-faces along $(d-1)$ faces, then for every $r \geqslant 0$, $S^r(\square)$ can be identified as a direct summand of $S^r(\square')$.

COROLLARY 3.1. *Let σ, σ' be any two d-faces of a d-complex $\Box$ and $\sigma=\sigma_0, \sigma_1, ..., \sigma_n=\sigma'$ be a chain of d-faces of $\Box$ such that $\forall i=0, 1, ..., n-1$, $\sigma_i \cap \sigma_{i+1}$ is a $(d-1)$-face. Then $S^r(\sigma_0 \cup \cdots \cup \sigma_n)$ is free. In particular, for any connected (hence any) 1-complex $\Box$ in $\mathbf{R}$, $S^r(\Box)$ is free.*

COROLLARY 3.2. *For any d-face σ of $\Box$, let $\Box'=\bigcup\{St\tau \mid \tau$ is a $(d-1)$-face of $\sigma\}$. Then for every $r \geqslant 0$, $S^r(\Box')$ is free.*

We may observe that the above corollaries represent respectively the purely vertical and purely horizontal attachings along $(d-1)$-faces. More importantly, one can easily see that in each of the above cases or even in the case when both are combined, one can write down an R-basis for the spline modules $S^r(\Box)$ in terms of the powers of linear forms defining interior $(d-1)$-faces. Referee informs that corollaries 3.1 and 3.2 also follow from [13, Theorem 5.1.].

For attachings along $(d-k)$-faces, $1 \leqslant k \leqslant d$ and for continuous splines we have

PROPOSITION 3.2. *Let σ_1, σ_2 be any two d-dimensional convex polyhedra such that $\tau=\sigma_1 \cap \sigma_2$ is a $(d-k)$-dimensional face of both. Then for the d-complex $\Box=\sigma_1 \cup \sigma_2$, we have*

$$pd_R S^0(\Box)=k-1.$$

Proof. *Since* $\dim \tau=d-k$, *the affine space $I(aff\ \tau)$ is generated by k independent affine forms, say $l_1, ..., l_k$. Now define the R-linear map $\varphi: S^0(\Box) \to I(\tau)$ by $\varphi(f_1, f_2)=f_1-f_2$. Then φ is well defined and admits an R-splitting $\psi: I(\tau) \to S^0(\Box)$ defined by $\psi(f)=(0, f)$. Hence, as R-modules $S^0(\Box) \cong Ker\varphi \oplus I(\tau)$. Because $I(\tau)=\langle l_1, ..., l_k\rangle$ is generated by k independent forms, $pd_R I(\tau)=k-1$. Also, $Ker(\varphi)=\{(f, f) \mid f \in R\}$ is a free R-module, which says that $pd_R S^0(\Box)=pd_R I(\tau)=k-1$.* ∎

In fact, the above result can be extended to general subcomplexes as follows:

PROPOSITION 3.3. *Let $\Box'$ be a d-subcomplex of a d-complex $\Box$ and suppose a d-face of $\Box$ is attached to a d-face σ' of $\Box'$ so that $\sigma \cap \sigma'=\tau$ is a $(d-k)$-face. Then for the d-complex $\Box''=\Box' \cup \sigma$, we have*

$$pd_R S^0(\Box'')=\max\{pd_R S^0(\Box'), k-1\}.$$

Proof. Order the d-faces of $\Box''$ so that σ' and σ occur at the end and define the $\mathbf{R}$-linear map $\varphi: S^0(\Box'') \to I(\tau)$ by $\varphi(f_{\sigma_1}, ..., f_{\sigma'}, f_\sigma)=f_\sigma-f_{\sigma'}$. Then again one has an $\mathbf{R}$-splitting $\psi: I(\tau) \to S^0(\Box'')$ defined by

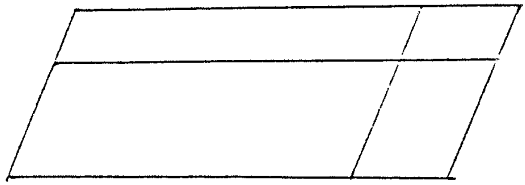

FIGURE 2

$\psi(f)=(0, 0, ..., 0, f)$ and so $S^0(\square'') \cong Ker\ \varphi \oplus I(\tau)$. One easily computes that $Ker\ \varphi \cong S^0(\square')$, and this completes the proof. ▮

If $S^r(\square)$ is free over R, then we know that $\square$ is hereditary [5]. The converse is not necessarily true unless $d=2$. However, for hereditary $\square$, $S^r(\square)$ is given by the kernel of an R-linear map between free R-modules (see [7]), for all $d \geqslant 1$. If $S^r(\square)$ is free, then the Gröbner basis method can be applied to write down a basis for $S^r(\square)$. In this connection, one comes across the following question: The entries of the matrix $A(\square, r)$ defining $S^r(\square)$ are $(r+1)$th powers of the linear forms defining the interior $(d-1)$ faces $\square$. Can we write down an R-basis for $S^r(\square)$ in terms of powers of these linear forms? When $d=1$, the answer is trivially "yes". For $d \geqslant 2$ and for polyhedral complexes very symmetrical from "inside', we can write down a basis for the free R-module $S^r(\square)$ in terms of the powers of linear forms defining the interior $(d-1)$ faces $\square$. First, we consider the case $d=2$ and have

LEMMA 3.1. *Suppose we have a parallelogram which has been divided into four smaller parallelograms by lines* $l_i=0$, $i=1, 2$. *Then for the resulting* 2-*complex* $\square$, *and for any* $r \geqslant 1$, *the set*

$$A=\{(1, 1, 1, 1), (0, \tilde{l}_1, \tilde{l}_1, 0)(0, 0, \tilde{l}_2, \tilde{l}_2)(0, 0, 0, \tilde{l}_1.\tilde{l}_2)\},$$

$$\tilde{l}_i=l_i^{r+1}, \qquad i=1, 2,$$

forms an R-*basis for* $S^r(\square)$ (*Fig.* 2).

Proof. We prove the case $r=0$ since for $r>0$, we have only to replace l_i by $\tilde{l}_i$ everywhere. Evidently, the set A is linearly independent over $R=\mathbf{R}[x_1, x_2]$ and so it suffices to show that $S^0(\square)$ is generated by the given elements, which are evidently in $S^0(\square)$. Let $s=(f_1, f_2, f_3, f_4) \in S^0(\square)$. By the algebraic criterion, we must have $g_1, g_2, g_3, g_4 \in R$ such that $f_2=f_1+g_1 l_1$, $f_3=f_1+g_1 l_1+g_2 l_2$, $f_4=f_1+g_1 l_1+g_2 l_2+g_3 l_1$, $f_1=f_1+g_1 l_1+g_2 l_2+g_3 l_1+g_4 l_2$. This means

$$l_1(g_1+g_3)=-l_2(g_2+g_4),$$

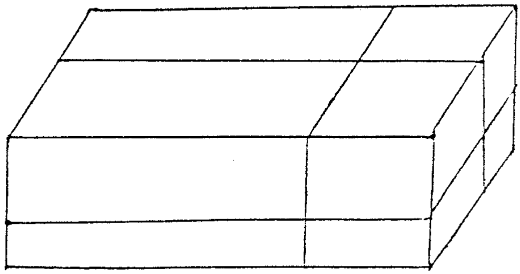

FIGURE 3

i.e., $g_1+g_3=h_3 l_2$ for some $h_3 \in R$. Hence

$$\begin{aligned} f_4 &= f_1+l_1 g_1+l_2 g_2+l_1(h_3 l_2-g_1) \\ &= f_1+l_2 g_2+h_3 l_1 l_2. \end{aligned}$$

Thus $s=f_1(1, 1, 1, 1)+g_1(0, l_1, l_1, 0)+g_2(0, 0, l_2, l_2)+h_3(0, 0, 0, l_1 . l_2)$. ∎

LEMMA 3.2. *Suppose we have a parallelopiped P in $\mathbf{R}^3$ which has been subdivided into 8 smaller parallelopipeds by hyperplanes $l_i=0$; $i=1, 2, 3$. Then for the resulting 3-complex $\square$ and for any $r \geqslant 0$, the set*

$$\begin{aligned} A=\{&(1, 1, 1, 1, 1, 1, 1, 1), (0, \tilde{l}_1, \tilde{l}_1, 0, 0, \tilde{l}_1, \tilde{l}_1, 0) \\ &(0, 0, \tilde{l}_2, \tilde{l}_2, 0, 0, \tilde{l}_2, \tilde{l}_2), (0, 0, 0, \tilde{l}_1 \tilde{l}_2, 0, 0, 0, \tilde{l}_1 \tilde{l}_2) \\ &(0, 0, 0, 0, \tilde{l}_3, \tilde{l}_3, \tilde{l}_3, \tilde{l}_3), (0, 0, 0, 0, 0, \tilde{l}_1 \tilde{l}_3, \tilde{l}_1 \tilde{l}_3, 0) \\ &(0, 0, 0, 0, 0, 0, \tilde{l}_2 \tilde{l}_3, l_2 \tilde{l}_3), (0, 0, 0, 0, 0, 0, 0, \tilde{l}_1 \tilde{l}_2 \tilde{l}_3)\} \end{aligned}$$

is a basis for $S^r(\square)$ for a suitable ordering of the maximal faces of $\square$ (Fig. 3).

Proof. We order the 3-faces of $\square$ so that the first four (anticlockwise) are above the plane $l_3=0$ and the fifth is below the first, sixth is below the second, etc. As in the proof of Lemma 3.1, we have only to prove the generating property and that too for the case $r=0$. Let $(f_1, ..., f_8) \in S^0(\square)$. By the previous Lemma and the algebraic criterion, we have $f_2=f_1+g_1 l_1$, $f_3=f_2+g_2 l_2$, $f_4=f_1+g_2 l_2+g_3 l_1 l_2$, $f_5=f_1+g_5 l_3$, $f_6=f_1+g_5 l_3+g_6 l_1=f_1+g_1 l_1+g_6' l_3$, for some g's. This means $g_6'-g_5=h_6 l_1$ some h_6, i.e.,

$$\begin{aligned} f_6 &= f_1+g_1 l_1+l_3(g_5+h_6 l_1) \\ &= f_1+g_1 l_1+g_5 l_3+h_6 l_1 l_3. \qquad (1) \end{aligned}$$

Also

$$f_7 = f_1 + g_1 l_1 + g_5 l_3 + h_6 l_1 l_3 + g_7 l_2$$
$$= f_1 + g_1 l_1 + g_2 l_2 + g'_7 l_3,$$

which means

$$(g'_7 - g_5 - h_6 l_1) l_3 = (g_7 - g_2) l_2$$

so that $g'_7 = g_5 + h_6 l_1 + k_7 l_2$ for some k_7, and hence

$$f_7 = f_1 + g_1 l_1 + g_2 l_2 + g_5 l_3 + h_6 l_1 l_3 + k_7 l_2 l_3. \tag{2}$$

Finally, we must have

$$f_8 = f_1 + g_1 l_1 + g_2 l_2 + g_5 l_3 + h_6 l_1 l_3 + k_7 l_2 l_3 + g'_8 l_1$$
$$= f_1 + g_2 l_2 + g_3 l_1 l_2 + h_7 l_3$$
$$= f_1 + g_5 l_3 + k_8 l_2,$$

for some g'_8, h_8 and $k_8 \in R$.

The last equality implies

$$h_7 - g_5 = k'_8 l_2,$$

for some k'_8 so now the second equality for f_8 becomes

$$f_8 = f_1 + g_2 l_2 + g_3 l_1 l_2 + g_5 l_3 + k'_8 l_2 l_3.$$

Then the first equality considered with above means

$$k'_8 - k_7 = h_8 l_1,$$

for some h_8, i.e.,

$$f_8 = f_1 + g_2 l_2 + g_5 l_3 + g_3 l_1 l_2 + k_7 l_2 l_3 + h_8 l_1 l_2 l_3. \tag{3}$$

This completes our computation, and expressions for f_i, $i = 2, 3, 4, 5$ along with (3.1) to (3.3) show that $(f_1, ..., f_8)$ is a linear combination of the given elements of $S_0(\square)$ with coefficients as $f_1, g_1, g_2, g_3, g_5, h_6, k_7$, and h_8. ∎

Now from the above two results and induction on d, one could prove the following:

PROPOSITION 3.4. *Let $P \subset \mathbf{R}^d$ $(d \geqslant 1)$ be a parallelopiped. Suppose P is subdivided in 2^d smaller parallelopipeds by hyperplanes $l_i = 0$; $i = 1, 2, ..., d$ parallel to each pair of parallel faces of P. Then there is an algorithm for* *ordering the d-faces of the resulting d-complex $\square$ and writing an explicit basis for the R-module $S^r(\square)$ in terms of the linear forms l_i and their products.*

The proof is omitted for reasons of lengthy details. However, the algorithm can be easily figured out from the two Lemmas. We must point out here that the boundary of the d-complexes in above results is totally immaterial—it could be even curved surfaces, polyhedra, etc. In other words, each member of $\square$ is a parallelopiped only from "inside" and bounding $(d-1)$-faces could be arbitrary. In particular, the corners of genuine parallelopipeds could be chopped off giving polyhedra which are not necessarily cubes and yet the results are all valid. (The referee points out that for the case $r=0$, Lemmas 3.1, 3.2 and Proposition 3.4 can also be derived from [3], Corollary 4.10; moreover, if one wants a simplicial situation, then a regular octahedron with an additional vertex at the centre can also be used to prove the above results.)

In Proposition 3.4, we do not have to take a $\square$ in which there are only 2^d faces. In fact, the results (3.1) (3.2) and (3.4) are given to show that if $\square$ is reasonably symmetric then we can attach d-faces of $\square$ not only along one $(d-1)$ face but along two, three or even more (at most d), $(d-1)$ faces simultaneously without increasing the projective dimension. For instance, if we partition a region Ω of $\mathbf{R}^2$ by drawing two sets of parallel lines crossing one another at any angle, then for the resulting 2-complex $\square$, it follows from (3.1) that for $r \geqslant 0$, $S^r(\square)$ is free.

Let us note that for any d-complex $\square \subset \mathbf{R}^d$, the R-module $S^r(\square)$ is a submodule of the free R-module R^t where $t = \#$ of d-faces in $\square$. Thus there is a short exact sequence of R-modules

$$0 \to S^r(\square) \to R^t \to N \to 0,$$

where N is the obvious quotient module. We know that $pd_R N \leqslant \dim R = d$. The above exact sequence shows that $S^r(\square)$ is the first syzygy of N. Since any two first syzygies of N are projectively equivalent, we get

Remark 3.2. For any d-complex $\square$ and any $r \geqslant 0$, $pd_R S^r(\square) \leqslant d-1$.

Now, we show that given any n, $0 \leqslant n \leqslant d-1$, there are plenty of d-complexes $\square$ embedded in $\mathbf{R}^d$ such that $pd_R S^r(\square) = n$. Actually, we give a d-complex $\square$ such that $S^0(\square)$ is free over R and has d-subcomplexes $\square^{(n)}$ such that $pd_R S^0(\square^{(n)}) = n$ for any n, $0 \leqslant n \leqslant d-1$.

EXAMPLE 3.1. Consider the d-complex $\square$ in $\mathbf{R}^d$ $(d \geqslant 2)$ consisting of 2^d generalized unit cubes formed by the intersections of hyperplanes

$x_i = 0, 1, -1$; $i = 1, 2, ..., d$. By (3.4) $S^0(\Box)$ is free over $R = \mathbf{R}[x_1, ..., x_d]$. Now, let

$$\sigma = \{(x_1, ..., x_d) \in \Box \mid x_i \geqslant 0 \ \forall i\}$$

and put

$$\sigma_1 = \{(x_1, ..., x_d) \in \Box \mid x_i \leqslant 0 \ \forall i\}.$$

Then since σ, σ_1 have just one vertex in common, (3.3) implies that $pd_R S^0(\Box^{(1)}) = d-1$ where $\Box^{(1)} = \sigma_1 \cup \sigma$. Next, let

$$\sigma_2 = \{(x_1, ..., x_d) \in \Box \mid x_i \leqslant 0 \ \forall i < d \text{ but } x_d \geqslant 0\}.$$

Then σ and σ_2 have a 1-dimensional face in common and so for $\Box^{(2)} = \sigma \cup \sigma_2$, $pd_R S^0(\Box^{(2)}) = d-2$. We can continue like this, and finally define

$$\sigma_d = \{(x_1, ..., x_d) \in \Box \mid x_i \leqslant 0, x_j \geqslant 0 \ \forall j \geqslant 2\}.$$

Then as before, we find that for $\Box^{(d)} = \sigma \cup \sigma_d$, $pd_R S^0(\Box^{(d)}) = 0$. Thus we have a d-complex $\Box$ such that $pd_R S^0(\Box) = 0$ but $\Box$ has d-subcomplexes which are of projective dimension n for any n from zero through $d-1$.

The zero-dimensional case of the following subset theorem for projective dimension of $S^r(\Box)$ has been proved in [5]. Note that star of a face of d-complex is a d-subcomplex.

THEOREM 3.1.[1] *Let $\Box$ be a d-complex embedded in $\mathbf{R}^d$ and $\Box' = St_{\Box}(\sigma)$ for some $\sigma \in \Box$. Then for each $r \geqslant 0$, $pd_R S^r(\Box') \leqslant pd_R S^r(\Box)$.*

Proof. The proof is an adaptation of the one given by Billera and Rose with appropriate modifications. Let $pd_R S^r(\Box) = n$ and v be a vertex of $\Box$. Let $P = I(v)$ be the maximal ideal of R consisting of all polynomials which vanish at v. Then using notations from [5] and as proved there, we find that

$$S^r(\Box)_p \cong [S^r(Stv) \oplus M]_p$$

as R_p-modules. Now by Corollary 2.1, $pd_{R_p} S^r(\Box)_p \leqslant n$ and hence $pd_{R_p} S^r(Stv)_p \leqslant pd_{R_p} S^r(\Box)_p \leqslant n$. Because $P = I(v)$, we conclude that $pd_R S^r(Stv) \leqslant n$. Next, let $\sigma \in \Box$ and $v_1, ..., v_k$ be the vertices of σ. Putting $\sum_1 = St_{\Box} v_i$ and $\sum_{i+1} = St_{\sum_i}(v_{i+1})$ for every $i = 1, ..., k-1$, and applying

[1] The referees inform us that this result has been obtained independently by L. Rose as well [14, Prop. 1.4].

the result proved about stars of vertices, we conclude that $pd_R S^r(\sum_k) \leqslant n$. Since $\sum_k = St_{\square}(\sigma)$, we find that for each $r \geqslant 0$

$$pd_R S^r(\square') \leqslant pd_R S^r(\square). \quad \blacksquare$$

Now we have our main result.

THEOREM 3.2. *Let $\square'$ be a d-subcomplex of a d-complex $\square$, and suppose $\square'$ has been obtained from the star of a face τ of $\square$ by attaching a finite sequence of d-face of $\square$ along $(d-1)$-face. Then for every $r \geqslant 0$,*

$$pd_R S^r(\square') \leqslant pd_R S^r(\square).$$

Proof. The proof follows immediately from Theorem 3.1 and Proposition 3.2. ∎

It may be observed that Theorem 3.2 is best possible in the sense that if we attach d-faces along k faces, $k < d-1$, then the result is not valid (Proposition 3.2).

A comparison. Let $\square$ be a polyhedral d-complex. For each $\sigma \in \square_d$, consider the affine subspaces $\mathrm{Aff}(\sigma \cap \tau)$ where $\tau \in \square_d$ and $\sigma \cap \tau \neq \varphi$. Let L_σ denote the set of all possible nonempty intersections of these subspaces of $\mathbf{R}^d$. Define a partial order on L_σ by reversing the inclusion relation. Next, let $P = \{(X, \sigma) \mid X \in L_\sigma \text{ and } \sigma \in \square_d\}$, and identify (X, σ) with (X, τ) if $X \subset \mathrm{Aff}(\sigma \cap \tau)$. Consider the equivalence relation on P generated by above identification, and let $L(\square)$ denote the set of all equivalence classes $(\overline{X, \sigma})$. Then $L(\square)$, which we will denote by L only, is easily seen to be a poset with the ordering defined by $(\overline{X, \sigma}) < (\overline{Y, \tau})$ if $(\overline{Y, \tau}) = (\overline{Y, \sigma})$ and $X < Y$ in L_σ. Then L is called the canonically associated poset to the given polyhedral complex $\square$.

For each $r \geqslant 0$, we now define a sheaf of R-modules on L (see [4] for details) as follows: to each $(\overline{X, \sigma})$ associate the R-module $R/(I(X))^{r+1}$, and to each pair $(\overline{X, \sigma}) < (\overline{Y, \tau})$ of related points, assigned the natural quotient R-homomorphism

$$R/(I(X))^{r+1} \to R/(I(Y))^{r+1}.$$

Let F^r denote the resulting sheaf of R-modules on L. Then, it is proved in [12] that for each $r \geqslant 0$, the R-module of sections of the sheaf F^r is isomorphic to the spline module $S^r(\square)$.

Finally, for each $x \in L$, let $\square_x$ denote the polyhedral subcomplex of $\square$ formed by all faces σ of $\square$ such that $(\overline{X, \sigma}) = x$ for some subspace X of $\mathbf{R}^d$. Now putting $S^r_x = S^r(\square_x)$, one has the following result [12].

PROPOSITION 3.5. (Yuzvinsky). *With above notations, one has*

(a) $pd_R S^r(\Box) \geqslant pd_R S^r_x$ *for every* $x \in L$.

(b) $pd_R S^r(\Box) = \max\{pd_R S^r_x\}$ *where* x *runs over all maximal elements of the poset* L.

Now we may like to compare Theorem 3.1 proved earlier and (a) of the above result. Both are subset Theorems for projective dimension of spline modules, but these are not related. For one thing, $S^r(St\sigma)$ need not always be of the form $S_x = S(\triangle_x)$ for any $x \in L$. Secondly, $\triangle_x$ might be larger than $St\sigma$ for any $\sigma \in \Box$; it may be union of more than one such stars. Combining the latter subset theorem with Proposition 3.1, we get a result similar to Theorem 3.2. Thus there is a large class of d-subcomplexes of a d-complex for which the projective dimension of spline modules respects the subset theorem.

4. SUM THEOREMS FOR PROJECTIVE DIMENSION; LOCAL NATURE

Following the sum theorems of classical dimension functions of topology, on can consider analogous results for projective dimension of spline modules $S^r(\Box)$ as follows: Let $\Box_i$, $i = 1, 2, ..., k$ be a finite collection of d-subcomplexes of $\Box$ such that $\Box = \bigcup \Box_i$. Then we will say that sum theorem holds for the projective dimension of spline modules for the collection $\{\Box_i\}$ if

$$pd_R S^r(\Box) = Sup\{pd_R S^r(\Box_i)\}.$$

A sum theorem when both sides of the above equality are zero has been proved ([5], Corollary 2.4), and another sum Theorem of general nature has been proved in ([12], Proposition 2.4(b)). Recall that for any $i \geqslant 0$, $\triangle_i$ denotes the set of all i-dimensional faces of $\triangle$. Then we have

THEOREM 4.1. *Let* $\triangle$ *be a simplicial complex embedded in* $\mathbf{R}^d$. *Then for any* $r \geqslant 0$,

$$\begin{aligned} pd_R S^r(\triangle) &= Sup_{\sigma \in \triangle}\{pd_R S^r(St\sigma)\} \\ &= Sup_{v \in \triangle_0}\{pd_R S^r(Stv)\}. \end{aligned}$$

Proof. Let $n = Sup_{\sigma \in \triangle}\{pd_R S^r(St\sigma)\}$. By the subset Theorem 3.1, we have only to show $pd_R S^r(\triangle) \leqslant n$. For this it is enough to show by

Corollary 2.1, that for each prime ideal p of R, $pd_{R_p}(S^r(\triangle)_p) \leqslant n$. By the same result, we know that

$$Sup\{pd_{R_p}(S^r(St\sigma)_p)\} \leqslant n \tag{4}$$

for each prime ideal p of R. Define $S(P) = \{\tau \in \triangle \mid I(\tau) \subset P$ and τ is minimal with respect to this property$\}$. Then, as proved in [5],

$$S^r(\triangle)_p \cong \bigoplus_{r \in S(P)} (S^r(St\tau)_p) \tag{5}$$

as R_p-modules. From (4) and (5) it follows that $pd_{R_p}(S^r(\triangle)_p) \leqslant n$. This proves the first equality. For the second, we note that if v is any vertex of simplex τ, then $St_\triangle \tau = St_{St_\triangle v} \tau$. By the subset Theorem 3.1, this implies that

$$pd_R S^r(St_{\triangle\tau}) \leqslant pd_R S^r(St_\triangle v),$$

which completes the proof. ▮

Local nature. At this point, we must emphasize an important aspect of the foregoing result. Even though it is stated as a sum theorem, the fact that

$$pd_R S^r(\triangle) = Sup_{v \in \triangle_0}\{pd_R S^r(Stv)\}$$

tells us that the projective dimension of a spline module $S^r(\triangle)$, for a simplicial complex $\triangle$, is indeed a local concept, i.e., the projective dimension of the spline module $S^r(\triangle)$ is completely determined by the projective dimension of the spline modules of stars of vertices of $\triangle$. The free case ($pd_R = 0$) has already been highlighted in [5]. As a matter of fact, this viewpoint of the above theorem puts the projective dimension of spline modules in line with other dimension functions of topology, e.g., the covering dimension and the cohomological dimension which are well known to be of local nature (see [10]) for paracompact Hausdorff spaces. Here the stars of vertices are to be interpreted as basic neighbourhoods for the d-complex $\triangle$. On the other hand the result of Yuzvinsky, Proposition 3.5(b), is valid for any polyhedral complex $\square$, not necessarily simplicial, but it does not assert the local nature of the projective dimension of $S^r(\square)$. In fact, it is not. We give an example later to show that for polyhedral complexes the projective dimension of the spline module of $\square$ is not determined by the projective dimension of spline modules of stars of vertices. The focus of Yuzvinsky's result is to show that for the study of projective dimension of $S^r(\square)$, it is enough to work with central complexes in which case $S^r(\square)$ becomes a graded R-module and one can apply methods of graded commutative algebra. Interpreted as a sum theorem, however, the collection $\{\Box_x \mid x \in \max L\}$ of d-subcomplexes of a d-complex $\Box$ in Proposition 3.5(b) may not consist of all stars of vertices or faces of $\Box$. The number in max$L$ may be less even in case of simplicial complexes. In other words, it is a distinct type of sum theorem as compared to Theorem 4.1.

Next we give an example of a polyhedral complex $\Box$ for which the spline module $S^r(\Box)$ is locally free over R but is not free.

EXAMPLE 4.1. Let us begin by examining for which dimension d we can find an example of a polyhedral complex $\Box$ such that $pd_R S^r(\Box)$ is not determined by the projective dimension of the spline modules of stars of vertices. In the one-dimensional case polyhedral complexes are identical with the simplicial complexes and so by Theorem 4.1, $pd_R S^r(\Box)$ is a local concept and is always zero. If $d=2$, then by the result of Billera and Rose $S^r(\Box)$ is free iff $\Box$ is a 2-manifold for each $r \geqslant 0$. We remarked in Section 3 that the maximum value of $pd_R S^r(\Box)$ for $\Box \subseteq \mathbf{R}^d$ is $d-1$. From these two observations and the fact that $\Box$ is a manifold with boundary iff the star of each vertex of $\Box$ is a manifold with boundary, we conclude that $pd_R S^r(\Box)$ is again a local concept for $d=2$ also. Hence for an example where $pd_R S^r(\Box)$ is not determined locally, it is necessary that $\Box \subseteq \mathbf{R}^d$ where $d \geqslant 3$.

Let us consider a 2-complex in the plane $\mathbf{R}^2$ consisting of five tilted quadrilaterals bounded by two concentric pentagons as shown in Fig. 4. We want the lines l_i, $i=1, 2, 3, 4, 5$, supporting the interior edges of the 2-complex to be in general position. Let $\Box'$ be a cone over this 2-complex so that $\Box'$ is a 3-complex in $\mathbf{R}^3$, having five maximal faces. Then for each $r \geqslant 0$, Yuzvinsky has shown ([12], Example 4.1) that $pd_R S^r(\Box') = 1$.

Now cut $\Box'$ by plane parallel to the base and let $\Box$ be the frustum (lower portion) of the cone, i.e., the one which does not contain the vertex

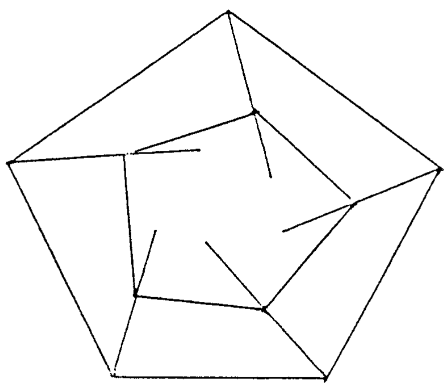

FIGURE 4

of $\square'$. Now consider the canonically associated posets $L(\square)$ and $L(\square')$ defined at the end of Section 3 and let F^r and $(F')^r$ denote the associated sheaves of R-modules on these posets. One can easily verify that (i) $L(\square) = L(\square')$ and (ii) $F^r = (F')^r$ for each $r \geqslant 0$. It follows. therefore, that

$$pd_R\, S^r(\square) = pd_R\, S^r(\square') = 1$$

for each $r \geqslant 0$. Hence the module $S^r(\square)$ is not free for any $r \geqslant 0$.

On the other hand, if v is any vertex of $\square$, then the star v consists of two 3-dimensional convex polyhedra with exactly one 2-dimensional face in common. Therefore, in view of Proposition 3.1, $pd_R\, S^r(Stv) = 0$ for every vertex v of $\square$. This proves that the module $S^r(\square)$ is locally free for every $r \geqslant 0$. ∎

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